

## Supplementary Material

### I. Proof of Theorem 1

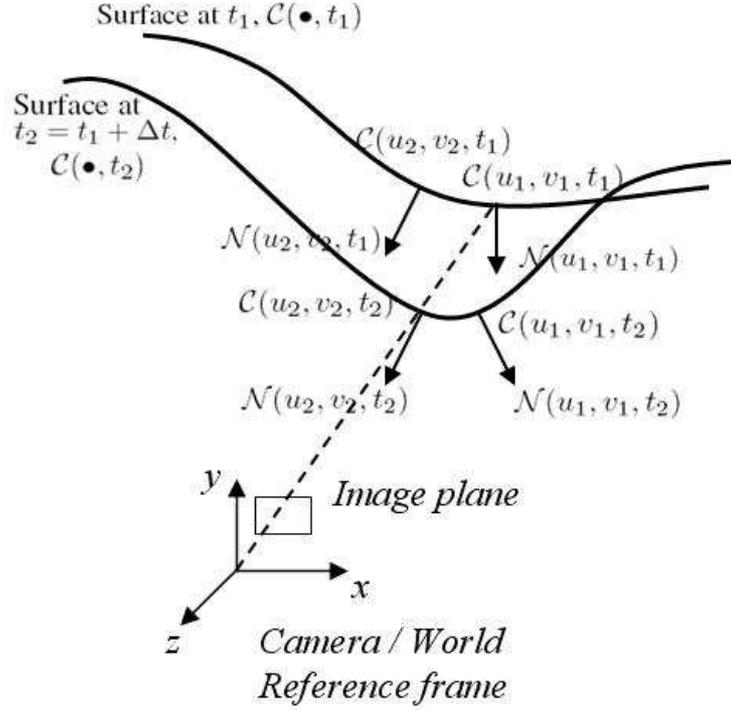


Figure 1: Pictorial representation depicting imaging framework.

We first define some notation required for our derivation. Let A and B represent the same object before and after deformation respectively, as shown in Fig. 1. The ray from the optical center to a particular pixel  $(x, y)$  intersects with the surface of the object at some point. Before the object's deformation, the ray intersects with the surface at  $\mathcal{C}(u_1, v_1, t_1)$  (on A), and after deformation, it intersects at  $\mathcal{C}(u_2, v_2, t_2)$  (on B). During the deformation,  $\mathcal{C}(u_2, v_2, t_1)$  (on A) evolves to  $\mathcal{C}(u_2, v_2, t_2)$  (on B). Note that  $\mathcal{C}(u_2, v_2, t_2)$  may not overlap with  $\mathcal{C}(u_1, v_1, t_1)$  - they are just on the same projection ray.

In [1], the authors show that, when a rigid object is fixed with respect to the camera, the reflectance image can be represented by a linear combination of a set of basis images. The basis images for each pixel can be expressed as

$$b_i(\mathcal{N}_j) = \rho_j r_i Y_i(\mathcal{N}_j), i = 0, 1, \dots, N_l \quad (1)$$

where  $\rho$  encrypts the surface reflectance property at the reflection point,  $Y_i$  is the spherical harmonics function, and  $r_i$  is a constant for each spherical harmonics order. Each  $b_i$  is the image appearance bases

vector of order  $i$ ,  $i = [0, N_l]$ , with a length of  $(P \times Q)$ , the number of pixels in the image. Assuming each  $b_i$  is a column vector, and concatenating  $N_l$  such bases column by column, we can form the  $\mathcal{B}_l$ , which is of size  $(P \times Q) \times N_l$ .

From (1), we see that when the illumination coefficients,  $l_i$ , are known, only the norm and the reflectance of the surface point of interest affect the reflection intensity at a particular pixel. The difference between  $\mathcal{N}(u_1, v_1, t_1)$  and  $\mathcal{N}(u_2, v_2, t_2)$  consists of two parts. The first part is the spatial change from  $\mathcal{N}(u_1, v_1, t_1)$  to  $\mathcal{N}(u_2, v_2, t_1)$ , while the second part is the temporal change due to the deformation from  $\mathcal{N}(u_2, v_2, t_1)$  to  $\mathcal{N}(u_2, v_2, t_2)$ . Both of these two parts are caused, and could be expressed analytically in terms of the motion and deformation parameters. The change of the texture, similarly, could be expressed in terms of the motion, deformation, and texture variation parameters too. Substitute all these variations into Eq. (1), we can have the analytical expression of the image appearance space in terms of these parameters. In our derivation, we will first show how to compute the spatial change, then combine it with the temporal change and finally the computation of the basis tensors which will lead to the proof of Theorem 1.

### Computation of Spatial Change Parameters

Let us now introduce a subscript  $w$  to denote the variables in the world reference frame. Since  $\mathcal{C}_w(u_1, v_1, t_1)$  and  $\mathcal{C}_w(u_2, v_2, t_2)$  are on the same ray (see Fig. 1), we can represent the difference between them using a unit vector  $\mathbf{r}$  under the perspective camera model as

$$\mathcal{C}_w(u_2, v_2, t_2) - \mathcal{C}_w(u_1, v_1, t_1) = k\mathbf{r}, \quad (2)$$

where  $k$  is a scalar. The transformation between the world frame and the object frame can be written as

$$\begin{aligned} \mathcal{C}_w(u_1, v_1, t_1) &= \mathbf{R}\mathcal{C}(u_1, v_1, t_1) + \mathbf{T}, \\ \mathcal{C}_w(u_2, v_2, t_2) &= \mathbf{\Delta R}\mathbf{R}\mathcal{C}(u_2, v_2, t_2) + \mathbf{\Delta T} + \mathbf{T}, \end{aligned} \quad (3)$$

where  $\mathbf{T}$  and  $\mathbf{R}$  are the translation and rotation matrix from the object reference frame to the world reference frame, while  $\mathbf{\Delta T}$  and  $\mathbf{\Delta R}$  are the translation and rotation matrix of the object in the world reference frame between  $t_1$  and  $t_2$ . Using the equations (3),(4) and (5) in the paper, the evolution of the object surface can be rewritten in a discrete format as

$$\mathcal{C}(u_2, v_2, t_2) = \mathcal{C}(u_2, v_2, t_1) + \mathbf{b}_d^{\mathbf{T}}(t_1)\Phi_d(u_2, v_2)\mathcal{N}(u_2, v_2, t_1)\Delta t. \quad (4)$$

Under Assumption (A1), which implies that the deformation between the two consecutive frames is small, the point  $\mathcal{C}(u_2, v_2, t_1)$  should be close to the point  $\mathcal{C}(u_1, v_1, t_1)$ . Thus, we may alternatively consider that the new point  $\mathcal{C}(u_2, v_2, t_1)$  is on the tangent plane that passes through the point  $\mathcal{C}(u_1, v_1, t_1)$ , i.e.,

$$\mathcal{C}(u_2, v_2, t_1) = \mathcal{C}(u_1, v_1, t_1) + (\mathcal{T}_u|_{u_1, v_1, t_1}, \mathcal{T}_v|_{u_1, v_1, t_1}) \mathbf{\Delta}, \quad (5)$$

where  $\mathcal{T}_u|_{u_1, v_1, t_1}$  represents the unit tangent vector along the direction of  $u$  at  $(u_1, v_1, t_1)$ , and  $\mathbf{\Delta}$  represents the difference in the surface parameters  $(u_1, v_1)$  and  $(u_2, v_2)$ . After a series of manipulations and using Assumption (A1) (see Section II of this document), we have

$$\mathbf{A}\mathbf{\Delta} = \left(\mathbf{I} - \frac{\mathbf{R}^{-1}\mathbf{r}\mathcal{N}^{\mathbf{T}}}{\mathcal{N}^{\mathbf{T}}\mathbf{R}^{-1}\mathbf{r}}\right)(\hat{\mathbf{C}}_1\mathbf{\Delta}\mathbf{\Omega} - \mathbf{R}^{-1}\mathbf{\Delta T} - \mathcal{N}\Phi_d^{\mathbf{T}}\mathbf{b}_d(t_1)\Delta t),$$

where

$$\mathbf{A} = \left(\mathbf{I} - \frac{\mathbf{R}^{-1}\mathbf{r}\mathcal{N}^{\mathbf{T}}}{\mathcal{N}^{\mathbf{T}}\mathbf{R}^{-1}\mathbf{r}}\right)(\mathbf{b}_d^{\mathbf{T}}(t_1)\Phi_d\mathbf{J}_N\Delta t + \mathcal{N}\mathbf{b}_d^{\mathbf{T}}(t_1)\nabla\Phi_d\Delta t) + (\mathcal{T}_u|_{t_1}, \mathcal{T}_v|_{t_1}), \quad (6)$$

and  $\hat{\mathbf{C}}_1$  denotes the skew symmetric matrix of vector  $\mathcal{C}(u_1, v_1, t_1)$ . Note that in (6),  $\mathcal{T}_u, \mathcal{T}_v, \mathcal{N}, \mathbf{J}_N, \mathbf{R}, \mathbf{r}$  are computed at  $t_1$  and  $\Phi_d, \nabla\Phi_d$  are constants in time. The first term  $\left(\mathbf{I} - \frac{\mathbf{R}^{-1}\mathbf{r}\mathcal{N}^{\mathbf{T}}}{\mathcal{N}^{\mathbf{T}}\mathbf{R}^{-1}\mathbf{r}}\right)(\mathbf{b}_d^{\mathbf{T}}\Phi_d\mathbf{J}_N\Delta t + \mathcal{N}\mathbf{b}_d^{\mathbf{T}}\nabla\Phi_d\Delta t) \sim O(\Delta t)$ , while the second term  $(\mathcal{T}_u|_{t_1}, \mathcal{T}_v|_{t_1}) \sim O(1)$ . Thus, using Assumption (A1) that  $\Delta t$  is small, the first term on the right hand side of the expression of  $\mathbf{A}$  in (6) can be ignored with respect to the second term. Consequently, the solution of  $\Delta$  can be written as

$$\begin{aligned} \Delta &= (\mathcal{T}_u, \mathcal{T}_v)^+ \left(\mathbf{I} - \frac{\mathbf{R}^{-1}\mathbf{r}\mathcal{N}^{\mathbf{T}}}{\mathcal{N}^{\mathbf{T}}\mathbf{R}^{-1}\mathbf{r}}\right)(\hat{\mathbf{C}}_1\Delta\Omega - \mathbf{R}^{-1}\Delta\mathbf{T} - \mathcal{N}\Phi_d^{\mathbf{T}}\mathbf{b}_d(t_1)\Delta t) \\ &\triangleq \mathbf{D}\Delta\Omega + \mathbf{E}\Delta\mathbf{T} + \mathbf{F}\mathbf{b}_d(t_1)\Delta t. \end{aligned} \quad (7)$$

and  $(\mathcal{T}_u, \mathcal{T}_v)^+$  indicates the pseudo inverse of the non-square matrix  $(\mathcal{T}_u, \mathcal{T}_v)$ .

### Computation of Basis Tensor

Next, we will compute the variation of the surface normal, and the variation of the basis tensor as a function of surface normal. As we have shown above, the difference between  $\mathcal{N}(u_1, v_1, t_1)$  and  $\mathcal{N}(u_2, v_2, t_2)$  consists of two parts. The first part is the spatial change from  $\mathcal{N}(u_1, v_1, t_1)$  to  $\mathcal{N}(u_2, v_2, t_1)$ , while the second part is the temporal change due to the deformation from  $\mathcal{N}(u_2, v_2, t_1)$  to  $\mathcal{N}(u_2, v_2, t_2)$ . Using Assumption (A1),  $\Delta t$  is small, thus the temporal change can be approximated with a first order Taylor expansion. Due to the same reason,  $\Delta$  should be a small term (in fact, in (7) we show that  $\Delta \sim O(\Delta t)$ ). Thus we can approximate the spatial change with a first order Taylor expansion at  $\mathcal{C}(u_1, v_1, t_1)$ . Thus we can express the change in norm as

$$\Delta\mathcal{N} = \mathcal{N}(u_2, v_2, t_2) - \mathcal{N}(u_1, v_1, t_1) = \mathbf{J}_N|_{u_1, v_1, t_1}\Delta + \frac{\partial\mathcal{N}(u_2, v_2, t)}{\partial t}|_{t_1}\Delta t, \quad (8)$$

where  $\mathbf{J}_N|_{u_1, v_1, t_1}$  is the Jacobian matrix of the norm,  $\mathcal{N}(u, v, t)$ , with respect to the parameters  $(u, v)$  at point  $\mathcal{C}(u_1, v_1, t_1)$ . The term  $\frac{\partial\mathcal{N}(u_2, v_2, t)}{\partial t}\Delta t$  is the temporal change of the  $\mathcal{N}(u_2, v_2)$ .

Similarly, for the texture change, we have

$$\begin{aligned} \rho(u_2, v_2, t_2) &= \Phi_\rho(u_2, v_2) \times_\rho \mathbf{b}_\rho^{\mathbf{T}}(t_2) \\ &= (\Phi_\rho(u_1, v_1) + \nabla\Phi_\rho|_{u_1, v_1}\Delta) \times_\rho \mathbf{b}_\rho^{\mathbf{T}}(t_2), \end{aligned} \quad (9)$$

Thus,  $\Delta\mathcal{N}$  and  $\rho(u_2, v_2, t_2)$  can be substituted into the expression for the basis images in (1), which can be rewritten as

$$\begin{aligned} b_i(u_2, v_2, t_2) &= ((\Phi_\rho(u_1, v_1) + \nabla\Phi_\rho|_{u_1, v_1}\Delta) \times_\rho \mathbf{b}_\rho(t_2))r_i Y_i(\mathcal{N}(u_1, v_1, t_1) + \Delta\mathcal{N}) \\ &= ((\Phi_\rho(u_1, v_1) + \nabla\Phi_\rho|_{u_1, v_1}\Delta)r_i Y_i(\mathcal{N}(u_1, v_1, t_1))) \\ &\quad + \Phi_\rho(u_1, v_1)r_i \nabla Y_i|_{\mathcal{N}(u_1, v_1, t_1)}\Delta\mathcal{N} \times_\rho \mathbf{b}_\rho(t_2) + O(\Delta^2). \end{aligned} \quad (10)$$

The last term is a higher order term of  $\Delta$ , which can be ignored.

When there exists both rigid motion and deformation, the temporal change of  $\mathcal{N}(u_2, v_2)$  from  $t_1$  to  $t_2$  consists of two parts: one due to the deformation, and one due to the rotation. In (8), using Assumption (A1) to neglect the terms  $O(\Delta t^2)$  with respect to  $O(\Delta t)$  and Assumption (A3) for smooth deformation

(see Section III of this document), we can derive the first part of the temporal change of norm from equation (3) in the paper purely due to deformation, i.e.,

$$\frac{\partial \mathcal{N}}{\partial t} \Big|_{u_2, v_2, t_1} \Delta t \Big|_{\Delta \Omega = 0} \approx - (\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u_1, v_1, t_1)) \mathbf{J}_{\mathcal{N}}(\Phi_d|(u_1, v_1)))^T \mathbf{b}_d(t_1) \Delta t. \quad (11)$$

Using assumption (A1), the second part of the temporal change due to the rigid rotation by  $\Delta \Omega$  is

$$\frac{\partial \mathcal{N}}{\partial t} \Big|_{u_2, v_2, t_1} \Delta t \Big|_{\mathbf{b}_d = 0} \approx - \hat{\mathcal{N}} \Big|_{u_1, v_1, t_1} \Delta \Omega. \quad (12)$$

Thus, substituting (11), (7), and (12) back into (8) and (9), the change of the norm and  $\rho$  can be expressed as

$$\begin{aligned} \Delta \mathcal{N} = & (\mathbf{J}_{\mathcal{N}}|_{u_1, v_1, t_1} \mathbf{D} - \hat{\mathcal{N}}|_{u_1, v_1, t_1}) \Delta \Omega + \mathbf{J}_{\mathcal{N}}|_{u_1, v_1, t_1} \mathbf{E} \Delta \mathbf{T} \\ & + (\mathbf{J}_{\mathcal{N}}|_{u_1, v_1, t_1} \mathbf{F} - \nabla \mathcal{C}|_{u_1, v_1, t_1} \nabla \Phi_d|_{u_1, v_1, t_1}^T) \mathbf{b}_d \Delta t. \end{aligned} \quad (13)$$

Thus, both  $\Delta \mathcal{N}$  and  $\Delta$  are linear functions of  $\Delta \mathbf{T}$ ,  $\Delta \Omega$  and  $\mathbf{b}_d$ . Substituting back into (10), and using tensor notation, we will have the equation (6) in the paper in Theorem 1.  $\square$

## II. Derivation of (6)

Substituting (5) and (3) into (2), we have

$$\Delta \mathbf{R}(\mathcal{C}(u_1, v_1, t_1) + \mathbf{b}_d^T \Phi_d(u_2, v_2) \mathcal{N}(u_2, v_2, t_1) \Delta t + (\mathcal{T}_u, \mathcal{T}_v) \Delta - \mathcal{C}(u_1, v_1, t_1)) = k \mathbf{R}^{-1} \mathbf{r} - \mathbf{R}^{-1} \Delta \mathbf{T}. \quad (14)$$

Using assumption (A1),  $\Delta$  should be small, thus we can apply Taylor expansion and have

$$\begin{aligned} \mathbf{b}_d^T \Phi_d(u_2, v_2) &= \mathbf{b}_d^T \Phi_d(u_1, v_1) + \mathbf{b}_d^T \nabla \Phi_d|_{u_1, v_1, t_1} \Delta, \\ \mathcal{N}(u_2, v_2, t_1) &= \mathcal{N}(u_1, v_1, t_1) + \mathbf{J}_{\mathcal{N}}|_{u_1, v_1} \Delta. \end{aligned} \quad (15)$$

Thus,  $\mathbf{b}_d^T \Phi_d(u_2, v_2) \mathcal{N}(u_2, v_2, t_1)$  can be expressed as

$$(\mathbf{b}_d^T \Phi_d + \mathbf{b}_d^T \nabla \Phi_d \Delta) (\mathcal{N} + \mathbf{J}_{\mathcal{N}} \Delta) = \mathbf{b}_d^T \Phi_d \mathcal{N} + \mathbf{b}_d^T \Phi_d \mathbf{J}_{\mathcal{N}} \Delta + \mathcal{N} \mathbf{b}_d^T \nabla \Phi_d \Delta + o(\Delta), \quad (16)$$

where all the terms are computed at  $(u_1, v_1, t_1)$ . Using reasoning similar to (16), the last term is a high order term thus can be ignored. Using (16) to approximate  $\mathbf{b}_d^T \Phi_d(u_2, v_2) \mathcal{N}(u_2, v_2, t_1)$ , we have

$$\begin{aligned} & (\Delta \mathbf{R} (\mathcal{T}_u, \mathcal{T}_v) + \mathbf{b}_d^T \Phi_d \Delta \mathbf{R} \mathbf{J} \Delta t + \Delta \mathbf{R} \mathcal{N} \mathbf{b}_d^T \nabla \Phi_d \Delta t) \Delta \\ &= (\mathbf{I} - \Delta \mathbf{R}) \mathcal{C}(u_1, v_1, t_1) - \mathbf{b}_d^T \Phi_d \Delta \mathbf{R} \mathcal{N} \Delta t - \mathbf{R}^{-1} \Delta \mathbf{T} + k \mathbf{R}^{-1} \mathbf{r}, \end{aligned} \quad (17)$$

where all the  $\mathcal{N}$ ,  $\mathbf{J}_{\mathcal{N}}$ ,  $\Phi_d$  and  $\nabla \Phi_d$  are at  $(u_1, v_1, t_1)$  and subscripts are discarded. Solving for  $k$ , we have

$$k \approx \frac{\mathcal{N}^T \mathbf{R}^{-1} \Delta \mathbf{T} + \mathcal{N}^T (\mathbf{I} - \Delta \mathbf{R}^{-1}) \mathcal{C}(u_1, v_1) + \mathbf{b}_d^T \Phi_d + (\mathbf{b}_d^T \Phi_d \mathcal{N}^T \mathbf{J}_{\mathcal{N}} + \mathbf{b}_d^T \nabla \Phi_d) \Delta}{\mathcal{N}^T \mathbf{R}^{-1} \mathbf{r}}. \quad (18)$$

Substituting back into (17), we have (6).

### III. Derivation of (11)

As

$$\mathcal{N} = \frac{\frac{\partial \mathcal{C}}{\partial u} \times \frac{\partial \mathcal{C}}{\partial v}}{\left\| \frac{\partial \mathcal{C}}{\partial u} \times \frac{\partial \mathcal{C}}{\partial v} \right\|} = \frac{\mathcal{C}_u \times \mathcal{C}_v}{\|\mathcal{C}_u \times \mathcal{C}_v\|} = \frac{\hat{\mathcal{C}}_u \mathcal{C}_v}{\sqrt{\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v}}, \quad (19)$$

where  $\hat{\mathcal{C}}$  denote the skew symmetric matrix with entries  $\begin{pmatrix} 0 & -\mathcal{C}^{(3)} & \mathcal{C}^{(2)} \\ \mathcal{C}^{(3)} & 0 & -\mathcal{C}^{(1)} \\ -\mathcal{C}^{(2)} & \mathcal{C}^{(1)} & 0 \end{pmatrix}$ , and the superscript

$\mathcal{C}^{(1)}$  indicates the first dimension of the vector. Taking the partial derivative of  $\mathcal{N}$  with respect to  $t$ , we have

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\frac{\partial \hat{\mathcal{C}}_u}{\partial t} \mathcal{C}_v + \hat{\mathcal{C}}_u \frac{\partial \mathcal{C}_v}{\partial t}}{\sqrt{\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v}} - \frac{\hat{\mathcal{C}}_u \mathcal{C}_v \frac{\partial (\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)}{\partial t}}{2(\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)^{\frac{3}{2}}}. \quad (20)$$

Taking the partial derivative of (4) in the paper with respect to  $u$  and  $v$ , and assuming  $\frac{\partial^2 \mathcal{C}}{\partial u \partial t}$  and  $\frac{\partial^2 \mathcal{C}}{\partial t \partial u}$  exist and are smooth (which is assumption **A3**), we have

$$\begin{aligned} \frac{\partial^2 \mathcal{C}}{\partial u \partial t} &= \frac{\partial \beta}{\partial u} \mathcal{N} + \beta \frac{\partial \mathcal{N}}{\partial u} = \beta_u \mathcal{N} + \beta \mathcal{N}_u = \frac{\partial \mathcal{C}_u}{\partial t}, \\ \frac{\partial^2 \mathcal{C}}{\partial v \partial t} &= \frac{\partial \beta}{\partial v} \mathcal{N} + \beta \frac{\partial \mathcal{N}}{\partial v} = \beta_v \mathcal{N} + \beta \mathcal{N}_v = \frac{\partial \mathcal{C}_v}{\partial t}. \end{aligned} \quad (21)$$

As the skew symmetric matrix  $\hat{\mathcal{C}}$  is linear with respect to the original vector  $\mathcal{C}$ , we have

$$\begin{aligned} \frac{\partial \hat{\mathcal{C}}_u}{\partial t} &= \beta_u \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_u, \\ \frac{\partial \hat{\mathcal{C}}_v}{\partial t} &= \beta_v \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_v. \end{aligned} \quad (22)$$

Substitute (21) and (22) back into the numerator of the first term in the right hand side of (20), we have

$$\begin{aligned} \frac{\partial \hat{\mathcal{C}}_u}{\partial t} \mathcal{C}_v + \hat{\mathcal{C}}_u \frac{\partial \mathcal{C}_v}{\partial t} &= (\beta_u \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_u) \mathcal{C}_v + \hat{\mathcal{C}}_u (\beta_v \mathcal{N} + \beta \mathcal{N}_v) \\ &= \beta_u \hat{\mathcal{N}} \mathcal{C}_v + \beta \hat{\mathcal{N}}_u \mathcal{C}_v + \hat{\mathcal{C}}_u \beta_v \mathcal{N} + \hat{\mathcal{C}}_u \beta \mathcal{N}_v \\ &= \beta_u \mathcal{N} \times \mathcal{C}_v + \beta \mathcal{N}_u \times \mathcal{C}_v + \beta_v \mathcal{C}_u \times \mathcal{N} + \beta \mathcal{C}_u \times \mathcal{N}_v. \end{aligned} \quad (23)$$

Similarly, the numerator of the second term in the right hand side of (20) can be simplified as

$$\begin{aligned} \frac{\partial (\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)}{\partial t} &= (\beta_v \mathcal{N}^{\mathbf{T}} + \beta \mathcal{N}_v^{\mathbf{T}}) \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v + (\beta_u \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}^{\mathbf{T}} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}_u^{\mathbf{T}}) \hat{\mathcal{C}}_u \mathcal{C}_v \\ &\quad + (\beta_u \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}}_u) \mathcal{C}_v + (\beta_v \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N}_v). \end{aligned} \quad (24)$$

Note that

$$\mathcal{N}^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v = (\mathcal{C}_u \times \mathcal{N})^{\mathbf{T}} (\mathcal{C}_u \times \mathcal{C}_v). \quad (25)$$

Because  $\mathcal{C}_u \parallel \mathcal{T}_u$  and  $\mathcal{C}_v \parallel \mathcal{T}_v$ , thus  $(\mathcal{C}_u \times \mathcal{N}) \perp \mathcal{N}$  while  $(\mathcal{C}_u \times \mathcal{C}_v) \parallel \mathcal{N}$ . Consequently, the inner product between the two terms in (25) is zero. Similarly, we have

$$\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v = \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}} \mathcal{C}_v = \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N} = 0. \quad (26)$$

Thus, (24) can be simplified as

$$\begin{aligned}
& \beta \mathcal{N}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}}_u \mathcal{C}_v + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N}_v \\
= & \beta (\mathcal{C}_u \times \mathcal{N}_v)^{\mathbf{T}} (\mathcal{C}_u \times \mathcal{C}_v) + \beta (\mathcal{N}_u \times \mathcal{C}_v)^{\mathbf{T}} (\mathcal{C}_u \times \mathcal{C}_v) \\
& + \beta (\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}} (\mathcal{N}_u \times \mathcal{C}_v) + \beta (\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}} (\mathcal{C}_u \times \mathcal{N}_v) \\
= & 2\beta (\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}} (\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v). \tag{27}
\end{aligned}$$

Thus, substituting (23) and (27) back into (20), we have

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\beta_u \mathcal{N} \times \mathcal{C}_v + \beta_v \mathcal{C}_u \times \mathcal{N}}{\|\mathcal{C}_u \times \mathcal{C}_v\|} + \beta \frac{\|\mathcal{C}_u \times \mathcal{C}_v\|^2 \mathbf{I} - (\mathcal{C}_u \times \mathcal{C}_v)(\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}}}{\|\mathcal{C}_u \times \mathcal{C}_v\|^3} (\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v). \tag{28}$$

Because  $\mathcal{N}_u \parallel \mathcal{C}_u$  and  $\mathcal{N}_v \parallel \mathcal{C}_v$ , thus  $(\mathcal{C}_u \times \mathcal{N}_v) \parallel (\mathcal{N}_u \times \mathcal{C}_v) \parallel \mathcal{N}$ . Let  $\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v = p\mathcal{N}$  and  $\mathcal{C}_u \times \mathcal{C}_v = q\mathcal{N}$ , where  $p$  and  $q$  are scalars. Thus the second term in the right hand side of (28) becomes

$$\beta \frac{q^2 p \mathcal{N} - q^2 \mathcal{N} \mathcal{N}^{\mathbf{T}} p \mathcal{N}}{q^3} = \beta \frac{q^2 p \mathcal{N} - q^2 p \mathcal{N}}{q^3} = 0. \tag{29}$$

Thus, (28) can be simplified as

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\beta_u \mathcal{N} \times \mathcal{C}_v + \beta_v \mathcal{C}_u \times \mathcal{N}}{\|\mathcal{C}_u \times \mathcal{C}_v\|}. \tag{30}$$

Thus, if  $\beta_u = 0$  and  $\beta_v = 0$ , the surface evolve isotropically, and the norm does not change over deformation. By choosing proper parameters  $u$  and  $v$ , we can let  $\|\mathcal{C}_u\| = 1$ ,  $\|\mathcal{C}_v\| = 1$ , and  $\mathcal{C}_u \perp \mathcal{C}_v$ . Use this set of parameterization and assume the right hand coordinate system to be  $(u \times v) \parallel \mathcal{N}$ , (30) can be simplified as

$$\frac{\partial \mathcal{N}}{\partial t} = -(\beta_u \mathcal{C}_u + \beta_v \mathcal{C}_v). \tag{31}$$

Thus, the second term in the right hand side of (8), i.e., temporal change of norm due to the deformation, can be simplified as

$$\begin{aligned}
\frac{\partial \mathcal{N}}{\partial t} \Big|_{u_2, v_2, t_1} \Delta t &= -(\mathbf{b}_d^{\mathbf{T}} \Phi_u \mathcal{C}_u + \mathbf{b}_d^{\mathbf{T}} \Phi_v \mathcal{C}_v) \Big|_{u_2, v_2, t_1} \\
&= -(\mathcal{C}_u, \mathcal{C}_v) \Big|_{u_2, v_2, t_1} \begin{pmatrix} \Phi_u^{\mathbf{T}} \\ \Phi_v^{\mathbf{T}} \end{pmatrix} \Big|_{u_2, v_2, t_1} \mathbf{b}_d \\
&= -\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \Big|_{(u_2, v_2, t_1)} \mathbf{J}_{\mathcal{N}}(\Phi|(u, v)) \Big|_{(u_2, v_2, t_1)}^{\mathbf{T}} \mathbf{b}_d. \tag{32}
\end{aligned}$$

Due to the fact that the change of the norm is not affected by the texture variation, for the simplicity of notation, we use  $\Phi$  to denote  $\Phi_d$  in the Appendix B. Substituting

$$\begin{aligned}
\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \Big|_{(u_2, v_2, t_1)} &= \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \Big|_{(u_1, v_1, t_1)} + \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v))}{\partial (u, v)} \Big|_{(u_1, v_1, t_1)} \times_3 \Delta, \\
\mathbf{J}_{\mathcal{N}}(\Phi|(u, v)) \Big|_{(u_2, v_2, t_1)} &= \mathbf{J}_{\mathcal{N}}(\Phi|(u, v)) \Big|_{(u_1, v_1, t_1)} + \frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))}{\partial (u, v)} \Big|_{(u_1, v_1, t_1)} \times_3 \Delta, \tag{33}
\end{aligned}$$

into (32), we have

$$\begin{aligned}
\frac{\partial \mathcal{N}}{\partial t} \Big|_{u_2, v_2, t_1} \Delta t &= -(\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) + \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v))}{\partial(u, v)} \times_3 \Delta) \\
&\quad \left( \mathbf{J}_{\mathcal{N}}(\Phi|(u, v)) + \frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))}{\partial(u, v)} \times_3 \Delta \right)^{\mathbf{T}} \mathbf{b} \\
&= -\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))^{\mathbf{T}} \mathbf{b}_d \\
&\quad - \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \left( \frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))}{\partial(u, v)} \times_3 \Delta \right)^{\mathbf{T}} \mathbf{b}_d \\
&\quad - \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v))}{\partial(u, v)} \times_3 \Delta \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))^{\mathbf{T}} \mathbf{b}_d \\
&\quad - \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v))}{\partial(u, v)} \times_3 \Delta \left( \frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))}{\partial(u, v)} \times_3 \Delta \right)^{\mathbf{T}} \mathbf{b}_d. \tag{34}
\end{aligned}$$

From (7), we know  $\Delta = O(\Delta t)$ . In addition, as  $\mathbf{b}_d = O(\Delta t)$ , the first term in the right hand side of (34) is  $O(\Delta t)$  while the other terms are  $O(\Delta t^2)$ . Using assumption **A2**, we can neglect  $O(\Delta t^2)$  with respect to  $O(\Delta t)$ , and (34) becomes,

$$\frac{\partial \mathcal{N}}{\partial t} \Big|_{u_2, v_2, t_1} \Delta t \approx -(\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u, v)) \mathbf{J}_{\mathcal{N}}(\Phi|(u, v))^{\mathbf{T}} \mathbf{b}_d. \tag{35}$$

## Globally multi-linear subspace

A piecewise multi-linear manifold can be embedded into a higher dimensional globally multi-linear subspace.

**Outline of the Proof:** Without loss of generality, we prove the case of piecewise bilinear manifold. Assuming we have a collection of locally bilinear manifold in the form of  $\mathcal{B}_j \times_1 \mathbf{a} \times_2 \mathbf{b}$ , where  $j$  is the indicator of the local manifold, and  $j = 1 \dots J$ . This piecewise manifold can be embedded into

$$\left( \begin{array}{cccc} \mathcal{B}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{B}_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{B}_J \end{array} \right) \times_1 \left( \begin{array}{c} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_J \end{array} \right) \times_2 \left( \begin{array}{c} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_J \end{array} \right), \tag{36}$$

where  $\mathbf{a}_1$  to  $\mathbf{a}_J$  and  $\mathbf{b}_1$  to  $\mathbf{b}_J$  are the same size of  $\mathbf{a}$  and  $\mathbf{b}$ . The  $j$ th piece of manifold can be obtained by setting all the  $\mathbf{a}$ s and  $\mathbf{b}$ s except  $\mathbf{a}_j$  and  $\mathbf{b}_j$  to be zero, while (36) forms a globally bilinear subspace.